

A PARTITION TEMPERLEY–LIEB ALGEBRA

(WORK IN PROGRESS)

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ABSTRACT. We introduce a generalization of the Temperley–Lieb algebra. This generalization is defined by adding certain relations to the algebra of braids and ties. A specialization of this last algebra corresponds to one small Ramified Partition algebra, this fact is the motivation for the name of our generalization.

INTRODUCTION

The Temperley–Lieb algebra appears originally in Statistical Mechanics as well as in Knot theory, quantum groups and subfactors of von Neumann algebras. This algebra was discovered by Temperley and Lieb by building transfer matrices[15]. Further, this algebra was rediscovered by V. Jones[8] who used it in the construction of his polynomial invariant for knots known as the Jones polynomial[9].

From a purely algebraic point of view, the Temperley–Lieb algebra is a quotient of the Iwahori–Hecke algebra by the two-sided ideal generated by the Steinberg elements h_{ij} associated to h_i 's, where $|i - j| = 1$ and h_i 's denote the usual generators of the Iwahori–Hecke algebra, view p. 35[5]. In other words, the Temperley–Lieb algebra can be defined by the usual presentation of the Iwahori–Hecke algebra but by adding the relations $h_{ij} = 0$, for all $|i - j| = 1$. Using this point of view, there are several generalizations of the Temperley–Lieb algebra, e.g. see [4, 6]. This paper proposes a generalization of the Temperley–Lieb algebra by adding relations of Steinberg types to the *algebra of braid and ties*[1, 14].

The algebra of braid and ties $\mathcal{E}_n(u)$, where u is a parameter and n denotes a positive integer, can be regarded as a generalization of the Hecke algebra and recently E. O. Banjo proved that $\mathcal{E}_n(1)$ is isomorphic to a small ramified partition algebra, see Theorem 4.2[2]. The possible connexion of the $\mathcal{E}_n(u)$ and the Partition algebras [10, 13] was speculated first by S. Ryom–Hansen[14]. The algebra $\mathcal{E}_n(u)$ is defined by two sets of generators and relations. One set of generators T_1, \dots, T_{n-1} reflects the braid generators of the Yokonuma–Hecke algebra[17, 16, 3] of type A and the other set of generators E_1, \dots, E_{n-1} reflects the behavior of the monoid P_n associated to the set partitions of $\{1, \dots, n\}$. Thus,

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$\mathcal{E}_n(u)$ also can be thought as a u -deformation of an amalgam among the symmetric group on n symbols and P_n .

In short, in this paper we define and study the *Partition Temperley–Lieb algebra*, denoted $\text{PTL}_n(u)$, which is defined by adding to the presentation of $\mathcal{E}_n(u)$ mentioned above the following relations

$$E_i E_j T_{ij} = 0 \quad \text{for all } |i - j| = 1$$

where T_{ij} is the Steinberg element associated to the T_i 's.

This work is organized as follows. In Section 1 we fix notations and we recall the definition of the Jimbo representation. In Section 2 we recall the definition of the algebra $\mathcal{E}_n(u)$, we have included also some results from [14] which are used in the paper. In Section 3 we construct a non-faithful tensor representation of the algebra $\mathcal{E}_n(u)$ which is used in Section 4 for the definition of our Partition Temperley–Lieb algebra $\text{PTL}_n(u)$. The Section 5 shows two presentations of the $\text{PTL}_n(u)$. By using one of these presentations we constructed a span linear set of $\text{PTL}_n(u)$ which is conjectured that is a basis for the Partition Temperley–Lieb algebra. Finally, based on a conjecture that the algebra $\mathcal{E}_n(u)$ supports a Markov trace, we prove in Section 7 under which condition this trace could pass to $\text{PTL}_n(u)$.

1. PRELIMINARIES

Along the paper algebra means unital associative algebra, with unity 1, over the field of rational function $K := \mathbb{C}(\sqrt{u})$ in the variable \sqrt{u} . Consequently, we put $u = (\sqrt{u})^2$.

Let $H_n = H_n(u)$ be the Iwahori–Hecke algebra of type A , that is, the algebra presented by generators $1, h_1, \dots, h_{n-1}$ subject to braid relations among the h_i 's and the quadratic relation $h_i^2 = u + (u - 1)h_i$, for all i .

We shall recall the Jimbo representation of the Hecke algebra. Set V the K -vector space with basis $\{v_1, v_2\}$. Denotes by \mathbf{J} the endomorphism of $V \otimes V$ defined through the mapping

$$\begin{aligned} \mathbf{J}(v_i \otimes v_j) &= -v_i \otimes v_j \quad \text{for } i = j \\ \mathbf{J}(v_1 \otimes v_2) &= (u - 1)v_1 \otimes v_2 + \sqrt{u}v_2 \otimes v_1 \\ \mathbf{J}(v_2 \otimes v_1) &= \sqrt{u}v_1 \otimes v_2. \end{aligned}$$

The Jimbo representation of H_n in $V^{\otimes n}$ is defined by mapping $h_i \mapsto \mathbf{J}_i$, where \mathbf{J}_i acts as the identity, with exception of the factors i and $i + 1$, where acts by \mathbf{J} .

Proposition 1.1. *The kernel of the Jimbo representation is the two-sided ideal generated by h_{ij} , where $|i - j| = 1$ and*

$$h_{ij} := 1 + h_i + h_j + h_i h_j + h_j h_i + h_i h_j h_i.$$

It is well known that the Temperley–Lieb algebra can be defined as the quotient of the Iwahori–Hecke algebra by the Kernel of Jimbo representation. Thus, the Temperley–Lieb algebra can be defined by adding extra non-redundant relations to the above presentations of the Hecke algebra. More precisely, we have the following definition.

Definition 1.2. *The Temperley-Lieb algebra $\text{TL}_n = \text{TL}_n(u)$ is the algebra generated by $1, h_1, \dots, h_{n-1}$ subject to the following relations:*

$$h_i^2 = u + (u-1)h_i \quad \text{for all } i \quad (1.1)$$

$$h_i h_j = h_j h_i \quad \text{for } |i-j| > 1 \quad (1.2)$$

$$h_i h_j h_i = h_j h_i h_j \quad \text{for } |i-j| = 1 \quad (1.3)$$

$$h_{ij} = 0 \quad \text{for } |i-j| = 1. \quad (1.4)$$

It is well known that the dimension of TL_n is the n th Catalan number $C_n := \frac{1}{n+1} \binom{2n}{n}$ [8] and that TL_n has a presentation (reduced) with idempotents generators. Indeed, making

$$f_i := \frac{1}{1+u}(1+h_i)$$

we have the following proposition.

Proposition 1.3. *TL_n can be presented by generators $1, f_1, \dots, f_{n-1}$ satisfying the following relations*

$$f_i^2 = f_i \quad \text{for all } i \quad (1.5)$$

$$f_i f_j = f_j f_i \quad \text{for } |i-j| > 1 \quad (1.6)$$

$$f_i f_j f_i = \frac{u}{(1+u)^2} f_i \quad \text{for } |i-j| = 1. \quad (1.7)$$

By virtue Proposition 1.1, the Jimbo representation of the Iwahori-Hecke algebra defines a representation of the Temperley-Lieb algebra. In terms of the generators f_i 's, this representation, called also the Jimbo representation, acts on $V^{\otimes n}$ by mapping $f_i \mapsto \mathbf{F}_i$. The endomorphism \mathbf{F}_i acts as the identity, with exception of the factors i and $i+1$, where acts by $\mathbf{F} \in \text{End}(V^{\otimes 2})$,

$$\begin{aligned} \mathbf{F}(v_i \otimes v_j) &= 0 \quad \text{for } i = j \\ \mathbf{F}(v_1 \otimes v_2) &= (u+1)^{-1}(u v_1 \otimes v_2 + \sqrt{u} v_2 \otimes v_1) \\ \mathbf{F}(v_2 \otimes v_1) &= (u+1)^{-1}(\sqrt{u} v_1 \otimes v_2 + v_2 \otimes v_1). \end{aligned}$$

2. THE ALGEBRA OF BRAIDS AND TIES

Let \mathbf{n} be the poset $\{1, \dots, n\}$. A partition of \mathbf{n} is a collection of pairwise disjoint subposets (called parts) whose union is equal to \mathbf{n} . We shall denote \mathbf{P}_n the set formed by the partitions of \mathbf{n} . The cardinal b_n of \mathbf{P}_n is known as the n th Bell number.

Let $I \in \mathbf{P}_n$, an arc $i \frown j$ of I is an ordered pair $(i, j) \in \{1, \dots, n\} \times \{1, \dots, n\}$ such that

- (1) $i < j$
- (2) i and j are in the same part of I
- (3) if k is in the same part as i and $i < k \leq j$, then $k = j$

In other words the arcs are pairs of adjacent elements in each part of I . Therefore the elements of \mathbf{P}_n can be represented by a graph with vertices $\{1, \dots, n\}$ and whose edge

connecting the vertices i and j if and only if $i \frown j$ is an arc of I . For example, for $n = 3$ we have

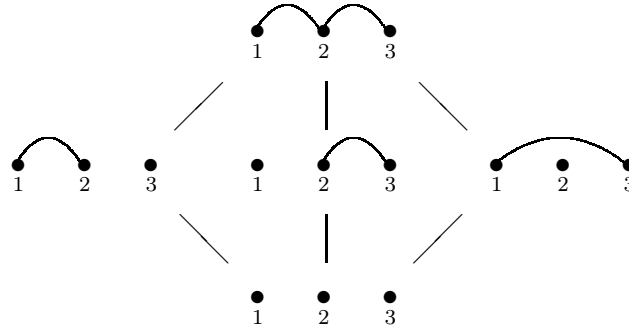
$$\{\{1, 2\}, \{3\}\} \quad \text{is represented by} \quad \begin{array}{ccc} \bullet & \text{---} & \bullet \\ 1 & & 2 \end{array} \quad \begin{array}{c} \bullet \\ 3 \end{array}$$

and so on.

The set P_n can be regarded naturally as a poset, where the partial order \preceq , is defined by: $I = (I_1, \dots, I_k) \preceq J = (J_1, \dots, J_l)$ if and only if each J_i is a union of certain I_i 's. By using \preceq we give to P_n a structure of commutative monoid by defining the product $I * J$, of I with J , as the minimum element of the poset P_n containing I and J . Clearly the unity is $\{\{1\}, \{2\}, \dots, \{n\}\}$ which is represented by $\begin{array}{cccc} \bullet & \bullet & \cdots & \bullet \\ 1 & 2 & & n \end{array}$. The monoid P_n is generated by the unity and the elements:

$$\begin{array}{ccccccc} \bullet & \cdots & \bullet & \text{---} & \bullet & \cdots & \bullet \\ 1 & & i & & i+1 & & n \end{array} \quad \text{for all } 1 \leq i \leq n$$

The Hasse diagram for P_3 is:



And we have, for example:

$$\begin{array}{ccc} \bullet & \text{---} & \bullet \\ 1 & & 2 \end{array} \quad \begin{array}{c} \bullet \\ 3 \end{array} * \begin{array}{c} \bullet \\ 1 \end{array} \quad \begin{array}{ccc} \bullet & \text{---} & \bullet \\ 2 & & 3 \end{array} = \begin{array}{ccc} \bullet & \text{---} & \bullet \\ 1 & & 2 \end{array} \quad \begin{array}{ccc} \bullet & \text{---} & \bullet \\ 2 & & 3 \end{array}$$

As usual we denote S_n the symmetric group on symbols and we denote s_i the transposition $(i, i + 1)$.

For $I = \{I_1, \dots, I_m\} \in P_n$ and $w \in S_n$ we define $wI = \{wI_1, \dots, wI_m\}$, where wI_i is the subposet of \mathbf{n} obtained by applying w to the elements of I_i .

Definition 2.1. We denote $\mathcal{E}_n = \mathcal{E}_n(u)$ the algebra generated by $1, T_1, \dots, T_{n-1}, E_1, \dots, E_{n-1}$ satisfying the following relations:

$$T_i T_j = T_j T_i \quad \text{for } |i - j| > 1 \quad (2.1)$$

$$T_i T_j T_i = T_j T_i T_i \quad \text{for } |i - j| = 1 \quad (2.2)$$

$$T_i^2 = 1 + (u - 1)E_i(1 + T_i) \quad \text{for all } i \quad (2.3)$$

$$E_i E_j = E_j E_i \quad \text{for all } i, j \quad (2.4)$$

$$E_i^2 = E_i \quad \text{for all } i \quad (2.5)$$

$$E_i T_j = T_j E_i \quad \text{for } |i - j| > 1 \quad (2.6)$$

$$E_i T_i = T_i E_i \quad \text{for all } i \quad (2.7)$$

$$E_i E_j T_i = T_i E_i E_j = E_j T_i E_j \quad \text{for } |i - j| = 1 \quad (2.8)$$

$$E_i T_j T_i = T_j T_i E_j \quad \text{for } |i - j| = 1. \quad (2.9)$$

If $w = s_{i_1} \cdots s_{i_k} \in S_n$ is reduced form for w , we write $T_w := T_{i_1} \cdots T_{i_k}$ (this is a possible debt to a well known result of H. Matsumoto).

For $i < j$, we define E_{ij} as

$$E_{ij} = \begin{cases} E_i & \text{for } j = i + 1 \\ T_i \cdots T_{j-2} E_{j-1} T_{j-2}^{-1} \cdots T_i^{-1} & \text{otherwise} \end{cases}$$

For any $J = \{i_1, i_2, \dots, i_k\}$ subposet of \mathbf{n} we define $E_J = 1$ if $k = 1$ and

$$E_J := E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_{k-1} i_k} \quad \text{for } k > 1$$

Note that $E_{\{i, j\}} = E_{ij}$. Also we note that in Lemma 4[14] it is proved that E_J can be computed as

$$E_J = \prod_{j \in J, j \neq i_0} E_{i_0 j} \quad (i_0 := \min\{i; i \in J\})$$

For $I = \{I_1, \dots, I_m\} \in \mathbf{P}_n$, we define E_I as

$$E_I = \prod_k E_{I_k}$$

The Corollary 2[14] implies the following proposition.

Proposition 2.2. *The mapping $E_i \mapsto \bullet \cdots \bullet \overset{i}{\curvearrowright} \bullet \cdots \bullet$ defines a monoid isomorphism between the monoid generated by $1, E_1, \dots, E_{n-1}$ and \mathbf{P}_n .*

Proposition 2.3 (Corollary 1[14]). *For $I \in \mathbf{P}_n$ and $w \in S_n$, we have*

$$T_w E_I T_w^{-1} = E_{wI}.$$

Theorem 2.4 (Corollary 3[14]). *The set $\mathbb{S}_n := \{E_I T_w; w \in S_n, I \in \mathbf{P}_n\}$ is a linear basis of \mathcal{E}_n . Hence the dimension of \mathcal{E}_n is $b_n n!$.*

3. A TENSORIAL REPRESENTATION FOR \mathcal{E}_n

In this section we will define a tensorial representation for \mathcal{E}_n . This representation is nothing more than a variation of that constructed by S. Ryom–Hansen in Section 3[14]. We note that, contrary to the representation constructed by Ryom–Hansen, our variation is a non-faithful representation. This fact is the key point in order to define the Partition Temperley–Lieb algebra as a quotient of \mathcal{E}_n .

Let V be the K -vector space with basis $\{v_i^r; 1 \leq i, r \leq n\}$, we define the endomorphisms \mathbf{E} and \mathbf{T} of $V^{\otimes 2}$ through the following mapping,

$$\mathbf{E}(v_i^r \otimes v_j^s) := \begin{cases} 0 & \text{for } r \neq s \\ v_i^r \otimes v_j^s & \text{for } r = s \end{cases}$$

$$\mathbf{T}(v_i^r \otimes v_j^s) := \begin{cases} -v_j^s \otimes v_i^r & \text{for } r \neq s \\ -v_i^r \otimes v_j^s & \text{for } r = s, i = j \\ (u-1)v_i^r \otimes v_j^s + \sqrt{u}v_j^s \otimes v_i^r & \text{for } r = s, i < j \\ \sqrt{u}v_j^s \otimes v_i^r & \text{for } r = s, i > j \end{cases}$$

Define now, \mathbf{E}_i (respectively \mathbf{T}_i) as the endomorphism of $V^{\otimes n}$ that acts as the identity with exception on the factors i and $i+1$ where acts by \mathbf{E} (respectively \mathbf{T}).

Theorem 3.1. *The mapping $E_i \mapsto \mathbf{E}_i$, $T_i \mapsto \mathbf{T}_i$ defines a representation \mathcal{J}_n of \mathcal{E}_n in $V^{\otimes n}$.*

Proof. The proof uses the same strategy as Theorem 1[14]. We only need to check that the operators \mathbf{E}_i and \mathbf{T}_i satisfy the respective relations (2.1)–(2.9). The relations (2.1), (2.4)–(2.7) clearly hold.

To check relation (2.3) it is enough to take $n = 2$. Evaluating the relation in $v_i^r \otimes v_j^s$ with $r = s$, the relation becomes the Hecke quadratic relation. In the case $r \neq s$, the operator $\mathbf{E}(1 + \mathbf{T})$ acts as zero and \mathbf{T}^2 as the identity, hence the relation holds.

To check the remaining of the relations, without loss of generality, we can suppose $n = 3$. Also we observe that it is enough to check the relations in question on the basis elements $x = v_i^r \otimes v_j^s \otimes v_k^t$. By simplicity we shall introduce the following notation: whenever we have two repetitions in the upper indices in the basis elements, we omit the two repeated upper indices and we replace the remaining indices by a prime, e.g. $v_i^r \otimes v_j^s \otimes v_k^r$ is written simply as $v_i \otimes v_j' \otimes v_k$. Then when we have two repetitions in the upper indices we shall distinguish three forms of elements:

$$v_i' \otimes v_j \otimes v_k \quad v_i \otimes v_j' \otimes v_k \quad v_i \otimes v_j \otimes v_k' \quad (3.1)$$

Further, in these elements we can suppose that the lower indices are 1 or 2 since \mathbf{T} acts according the order in the pair formed by lower indices. Now, the action of \mathbf{T} on primed and unprimed elements is, up to sign, a transposition, so we can suppose that the lower index of the primed factor is always 1. Therefore, the elements in the form as (3.1) can be reduced to consider the following cases:

$$\begin{array}{lll} v_1' \otimes v_1 \otimes v_1 & v_1 \otimes v_1' \otimes v_1 & v_1 \otimes v_1 \otimes v_1' \\ v_1' \otimes v_1 \otimes v_2 & v_1 \otimes v_1' \otimes v_2 & v_1 \otimes v_2 \otimes v_1' \\ v_1' \otimes v_2 \otimes v_1 & v_2 \otimes v_1' \otimes v_1 & v_2 \otimes v_1 \otimes v_1' \\ v_1' \otimes v_2 \otimes v_2 & v_2 \otimes v_1' \otimes v_2 & v_2 \otimes v_2 \otimes v_1' \end{array} \quad (3.2)$$

The checking of (2.8) and (2.9) are similar and routine. Thus we shall check only the first relation of (2.8). If all upper indices in x are distinct the operator $\mathbf{E}_i \mathbf{E}_j$ acts as zero and as the identity if all upper indices are equals. Hence $\mathbf{E}_1 \mathbf{E}_2 \mathbf{T}_1$ and $\mathbf{T}_1 \mathbf{E}_1 \mathbf{E}_2$ coincide on such x 's. Now it is easy to check the relation whenever x is an element of (3.2) whose unprimed factor has equal lower indices. The checking on the other elements of (3.2) results from a direct computation, e.g., for $x = v_1 \otimes v_2 \otimes v'_1$ we have

$$\mathbf{E}_1 \mathbf{E}_2 \mathbf{T}_1(x) = (u-1) \mathbf{E}_1 \mathbf{E}_2(x) + \sqrt{u} \mathbf{E}_1 \mathbf{E}_2(v_2 \otimes v_1 \otimes v'_1) = 0 = \mathbf{T}_1 \mathbf{E}_1 \mathbf{E}_2(x)$$

Finally we will check the relation (2.2). If in the basis elements the upper indices are all equal we are in the situation of Jimbo representation \mathbf{J} . If all upper indices are different the action becomes, up to sign, in the permutation action on the factors of the basis elements. Therefore, it only remains to check that (2.2) is true when one evaluates on the elements of (3.2). Now, it is easy to see that the evaluation of both sides of (2.2) on the elements of (3.2) whose unprimed factors are equal is $-\sigma_{13}$, where σ_{13} permutes the first with the third factor in the tensor product. The check of (2.2) on the remaining elements of (3.2) is all similar for all. We shall do, as a representative case, the case $x = v'_1 \otimes v_1 \otimes v_2$:

$$\begin{aligned} \mathbf{T}_2 \mathbf{T}_1 \mathbf{T}_2(x) &= (u-1) \mathbf{T}_2 \mathbf{T}_1(v'_1 \otimes v_1 \otimes v_2) + \sqrt{u} \mathbf{T}_2 \mathbf{T}_1(v'_1 \otimes v_2 \otimes v_1) \\ &= -(u-1) \mathbf{T}_2(v_1 \otimes v'_1 \otimes v_2) - \sqrt{u} \mathbf{T}_2(v_2 \otimes v'_1 \otimes v_1) \\ &= (u-1)(v_1 \otimes v_2 \otimes v'_1) + \sqrt{u}(v_2 \otimes v_1 \otimes v'_1) \\ &= \mathbf{T}_1(v_1 \otimes v_2 \otimes v'_1) \\ &= -\mathbf{T}_1 \mathbf{T}_2(v_1 \otimes v'_1 \otimes v_2) = \mathbf{T}_1 \mathbf{T}_2 \mathbf{T}_1(x). \end{aligned}$$

□

4. THE PTL algebra

We want to define a generalization of Temperley-Lieb algebra by using the algebra \mathcal{E}_n , we shall call this generalization the Partition Temperley-Lieb algebra which is denoted PTL_n . A first natural attempt of definition PTL_n is as the algebra that results by adding to defining relations of \mathcal{E}_n the relations $T_{ij} = 0$, where T_{ij} are the Steinberg elements T_{ij} 's associated to the T_i 's,

$$T_{ij} := 1 + T_i + T_j + T_i T_j + T_j T_i + T_i T_j T_j \quad \text{where} \quad |i - j| = 1$$

As in the classical case we want that the Jimbo representation \mathcal{J} of \mathcal{E}_n passes to PTL_n , hence the T_{ij} 's must be killed by \mathcal{J} . But unfortunately this does not happen. In fact, for $n = 3$ and by taking $x = v_1 \otimes v_2 \otimes v'_1$, we have

$$\begin{aligned} \mathbf{T}_1 x &= (u-1)v_1 \otimes v_2 \otimes v'_1 + \sqrt{u} v_2 \otimes v_1 \otimes v'_1 & \mathbf{T}_2 x &= -v_1 \otimes v'_1 \otimes v_2 \\ \mathbf{T}_2 \mathbf{T}_1 x &= -(u-1)v_1 \otimes v'_1 \otimes v_2 - \sqrt{u} v_2 \otimes v'_1 \otimes v_1 & \mathbf{T}_1 \mathbf{T}_2 x &= v'_1 \otimes v_1 \otimes v_2 \\ \mathbf{T}_1 \mathbf{T}_2 \mathbf{T}_1 x &= (u-1)v'_1 \otimes v_1 \otimes v_2 + \sqrt{u} v'_1 \otimes v_2 \otimes v_1 \end{aligned}$$

Then

$$\begin{aligned} (\mathcal{J} T_{12})x &= u v_1 \otimes v_2 \otimes v'_1 - u v_1 \otimes v'_1 \otimes v_2 + \sqrt{u} v_2 \otimes v_1 \otimes v'_1 \\ &\quad - \sqrt{u} v_2 \otimes v'_1 \otimes v_1 + u v'_1 \otimes v_1 \otimes v_2 + \sqrt{u} v'_1 \otimes v_2 \otimes v_1 \end{aligned}$$

Therefore \mathcal{J} does not kill T_{12} .

Having in mind the above discussion we make the following definition.

Definition 4.1. *The Partition Temperley–Lieb algebra $\text{PTL}_n = \text{PTL}_n(u)$ is defined by adding to the defining presentation of \mathcal{E}_n the relations:*

$$E_i E_j T_{i,j} = 0 \quad \text{for all } |i - j| = 1. \quad (4.1)$$

Clearly, from (2.8) we have that $E_i E_j T_{i,j} = 0$ is equivalent to $T_{i,j} E_i E_j = 0$.

Remark 4.2. Notice that by taking $E_i = 1$ the algebra PTL_n coincides with the classical Temperley–Lieb algebra. Also, we note that the defining relations of PTL_n hold when T_i is replaced by the generators h_i of the Temperley–Lieb algebra and E_i is replaced by 1, thus the mapping $E_i \mapsto 1$ and $T_i \mapsto h_i$ defines an algebra homomorphism from PTL_n onto TL_n .

Theorem 4.3. *The Jimbo representation \mathcal{J}_n of \mathcal{E}_n factors through the algebra PTL_n .*

Proof. Without loss of generality we can suppose that $n = 3$. Thus, we must prove that $\mathcal{J}_3(E_1 E_2 T_{12}) = 0$. Now, keeping the notations used during the proof of Theorem 3.1, to prove the theorem it is enough to see that $\mathcal{J}_3(E_1 E_2 T_{12})$ kill the basis elements $x = v_i^r \otimes v_j^s \otimes v_k^t$. If all upper indices in x are equal, \mathcal{J}_3 is the Jimbo representation of the Hecke algebra, so $\mathcal{J}_3(T_{12})$ kill x ; hence $\mathcal{J}_3(E_1 E_2 T_{12})$ kill x too. If the upper indices of x are not all equal, we have that x is killed by \mathbf{E}_1 or \mathbf{E}_2 , hence $\mathcal{J}_3(E_1 E_2 T_{12})(x) = 0$. \square

We are going to prove now that the set of relations (4.1) can be reduced to only one. To do this we need to introduce the following element Γ ,

$$\Gamma := T_1 T_2 \cdots T_{n-1}$$

Lemma 4.4. *For all $1 \leq i, j \leq n - 1$ we have:*

- (1) $T_i = \Gamma^{i-1} T_1 \Gamma^{-(i-1)}$
- (2) $T_{i,i+1} = \Gamma^{i-1} T_{1,2} \Gamma^{-(i-1)}$
- (3) $E_i = \Gamma^{i-1} E_1 \Gamma^{-(i-1)}$
- (4) $T_{i+1} \Gamma^{i-1} = \Gamma^{i-1} T_2$
- (5) $E_{\{i,i+2\}} = \Gamma^{i-1} E_{\{1,3\}} \Gamma^{-(i-1)}$

Proof. The statement (1) results from an inductive argument on i and the braid relations of T_i 's. The statement (2) is a result applying (1). The proof of statement (3) is analogous to the proof of (1), that is: an argument inductive on i and using the relation (2.6). The statement (4) is clear, since (1). Finally, we have:

$$\begin{aligned} \Gamma^{i-1} E_{\{1,3\}} \Gamma^{-(i-1)} &= \Gamma^{i-1} T_2 E_1 T_2^{-1} \Gamma^{-(i-1)} \\ &= \Gamma^{i-1} T_2 (\Gamma^{-(i-1)} E_i \Gamma^{(i-1)}) T_2^{-1} \Gamma^{-(i-1)} \\ &= T_{i+1} E_i T_{i+1}^{-1} \end{aligned}$$

Thus, the statement (5) is proved. \square

Proposition 4.5. *The relation $E_1 E_2 T_{1,2} = 0$ implies the relations $E_i E_j T_{i,j} = 0$, for all $|i - j| = 1$.*

Proof. We can suppose $j = i + 1$, since $T_{ij} = T_{ji}$ and E_i and E_j commute. From the statements (1) and (3) Lemma 4.4, we have:

$$\begin{aligned} E_i E_{i+1} T_{i,i+1} &= (\Gamma^{i-1} E_1 \Gamma^{-(i-1)}) (\Gamma^i E_1 \Gamma^{-i}) (\Gamma^{i-1} T_{1,2} \Gamma^{-(i-1)}) \\ &= \Gamma^{i-1} E_1 \Gamma E_1 \Gamma^{-1} T_{1,2} \Gamma^{-(i-1)} = \Gamma^{i-1} E_1 E_2 T_{1,2} \Gamma^{-(i-1)} \end{aligned}$$

Hence the proof follows. \square

Corollary 4.6. *The Partition Temperley-Lieb algebra PTL_n can be regarded as the quotient of \mathcal{E}_n by the two-sided ideal generated by $E_1 E_2 T_{12}$.*

5. OTHERS PRESENTATIONS FOR PTL_n

In order to have more comfortable notations we shall introduce the following element δ ,

$$\delta := \frac{1-u}{1+u} \in K$$

5.1. Having in mind the definition of the idempotents generators f_i of the Temperley-Lieb algebra, it is natural to consider the following definition.

$$F_i := \frac{1}{u+1} (1 + T_i) \quad (1 \leq i \leq n-1)$$

It is obvious that F_i commute with E_i (and T_i) and that they form a set of generators for the algebra PTL_n , but notice that the F_i 's are not idempotents. In fact, from (2.3) we have

$$F_i^2 = \frac{1}{(u+1)^2} (1 + 2T_i + 1 + (u-1)E_i + (u-1)E_i T_i)$$

then

$$F_i^2 = (1 + \delta)F_i - \delta E_i F_i$$

We have the following proposition

Theorem 5.1. *PTL_n can be presented by the generators $1, E_1, \dots, E_{n-1}, F_1, \dots, F_{n-1}$ subject to the relations (2.4), (2.5) together with the following relations*

$$F_i^2 = (1 + \delta)F_i - \delta E_i F_i \quad (5.1)$$

$$F_i F_j = F_j F_i \quad \text{for all } |i-j| > 1 \quad (5.2)$$

$$F_i E_j = E_j F_i \quad \text{for all } |i-j| > 1 \quad (5.3)$$

$$E_i F_i = F_i E_i \quad (5.4)$$

and for all $|i-j| = 1$:

$$E_i E_j F_i = F_i E_i E_j = E_j F_i E_j + \frac{1}{u+1} (E_i E_j - E_j) \quad (5.5)$$

$$E_i F_j F_i = F_j F_i E_j + \frac{1}{u+1} [(E_i - E_j)F_j + F_i(E_i - E_j)] - \frac{1}{(u+1)^2} (E_i - E_j) \quad (5.6)$$

$$F_i F_j F_i = \frac{1}{(u+1)^2} (F_i - (1-u)E_i F_i) \quad (5.7)$$

Proof. It is easy to check that (2.1) (respectively (2.6)) is equivalent to (5.2) (respectively (5.3)), so having in mind the previous discussion to the theorem, it only remains to prove that the relations (5.5)–(5.7) hold and that relations (2.8), (2.9), (4.1) and (2.2) can be deduced from the relations of the theorem.

We have that $T_i = (u+1)F_i - 1$. Now replacing this expression of T_i in (2.8) (respectively (2.9)) it is a routine to check that (2.8) becomes (5.5) (respectively (5.6)).

We have to check that relation (4.1) is equivalent to relation (5.7). We have

$$T_i T_j = ((u+1)F_i - 1)((u+1)F_j - 1) = (u+1)^2 F_i F_j - (u+1)F_i - (u+1)F_j + 1$$

then

$$\begin{aligned} T_i T_j T_i &= (u+1)^3 F_i F_j F_i - (u+1)^2 F_i^2 - (u+1)^2 F_j F_i + (u+1)F_i \\ &\quad - (u+1)^2 F_i F_j + (u+1)F_i + (u+1)F_j - 1 \end{aligned}$$

Therefore, by using (5.1), we deduce

$$\begin{aligned} T_i T_j T_i &= (u+1)^3 F_i F_j F_i + (1-u^2)E_i F_i \\ &\quad - (u+1)^2 F_j F_i - (u+1)^2 F_i F_j + (u+1)F_j - 1 \end{aligned}$$

Now, substituting each summand of T_{ij} by its expression in term of F_i 's one obtains

$$T_{ij} = (u+1)^3 F_i F_j F_i + (1-u^2)E_i F_i - (u+1)F_i$$

Hence (4.1) is equivalent (5.7).

Finally notice that (5.7) implies (2.2), since the above expression of $T_i T_j T_i$ in terms of F_i 's tells us that (2.2) is equivalent to

$$(u+1)^2 F_i F_j F_i + (1-u)E_i F_i + F_j = (u+1)^2 F_j F_i F_j + (1-u)E_j F_j + F_i$$

Thus the proof is concluded. \square

5.2. In this subsection we shall show a presentation of PTL_n by idempotent generators. For $1 \leq i < j \leq n-1$, we define

$$L_i := \frac{1}{1+u} (T_i + 1) (\alpha + (1-\alpha)E_i) \quad \text{where} \quad \alpha := \frac{1+u}{2}$$

notice that

$$L_i = \frac{1}{2} (T_i + \delta T_i E_i + \delta E_i + 1) = \frac{1}{2} (1 + T_i)(1 + \delta E_i) \quad (5.8)$$

Also we have

$$L_i = \frac{u+1}{2} F_i + \frac{1-u}{2} E_i F_i \quad (5.9)$$

It is clear that L_i commute with E_i , T_i and F_i . We have the following useful lemma.

Lemma 5.2. *For all i we have:*

- (1) $L_i^2 = L_i$
- (2) $(1+u)E_i L_i = E_i(1+T_i)$
- (3) $T_i = 2L_i + (u-1)E_i L_i - 1$
- (4) $E_i L_i = E_i F_i$
- (5) $F_i = (1+\delta)L_i - \delta E_i L_i$.

Proof. We have:

$$L_i^2 = 4^{-1}(1 + T_i)^2(1 + \delta E_i)^2 = 4^{-1}(2(1 + T_i) + (u - 1)E_i(1 + T_i))(1 + (2\delta + \delta^2)E_i)$$

then

$$\begin{aligned} L_i^2 &= 4^{-1}(1 + T_i)(2 + (u - 1)E_i)(1 + (2\delta + \delta^2)E_i) \\ &= 4^{-1}(1 + T_i)(2 + (2(2\delta + \delta^2) + (u - 1) + (u - 1)(2\delta + \delta^2))E_i) \\ &= 4^{-1}(1 + T_i)(2 + (2\delta + \delta^2)(1 + u) + u - 1)E_i) \\ &= 4^{-1}(1 + T_i)(2 + 2\delta E_i) = L_i. \end{aligned}$$

The second assertion follows by multiplying the formula of L_i by E_i . To prove the third assertion, we bring first $E_i T_i$ from the second assertion and then we substitute this expression of $E_i T_i$ in (5.8), thus the third assertion follows. The fourth assertion results by multiplying (5.9) by E_i . The fifth assertion result directly from (4) and (5.9). \square

Theorem 5.3. *PTL_n can be presented by the generators $1, E_1, \dots, E_{n-1}, L_1, \dots, L_{n-1}$ subject to the relations (2.4), (2.5) together with the following relations*

$$L_i^2 = L_i \tag{5.10}$$

$$L_i L_j = L_j L_i \quad \text{for all } |i - j| > 1 \tag{5.11}$$

$$L_i E_j = E_j L_i \quad \text{for all } |i - j| > 1 \tag{5.12}$$

$$L_i E_i = E_i L_i \tag{5.13}$$

and for all $|i - j| = 1$:

$$E_i E_j L_i = L_i E_i E_j = E_j L_i E_j + 2^{-1}(E_i E_j - E_j) \tag{5.14}$$

$$4L_i L_j E_i + 2E_j(L_j + L_i) + E_i = 4E_j L_i L_j + 2(L_i + L_j)E_i + E_j \tag{5.15}$$

$$\begin{aligned} &8L_i L_j L_i + 4(u - 1)[L_i E_j L_j L_i + E_i L_i L_j L_i + L_i L_j E_i L_i] \\ &+ (u - 1)^2(u + 5)E_i E_j L_i L_j L_i = 2L_i + 3(u - 1)E_i L_i + (u - 1)^2 E_i E_j L_i \end{aligned} \tag{5.16}$$

Proof. We will use the presentation of Theorem 5.1. From (5)Lemma 5.2, follows that PTL_n is generated by 1, E_i 's and L_i 's. Checking that (5.1)–(5.6) are equivalent, respectively, to (5.10)–(5.15) is a straight forward and just a routine, so we leave the computation to the reader. Thus, to finish the proof it only remains to check that (5.16) is equivalent to (5.7).

We have

$$\begin{aligned} F_i F_j &= ((1 + \delta)L_i - \delta E_i L_i)((1 + \delta)L_j - \delta E_j L_j) \\ &= (1 + \delta)^2 L_i L_j - \delta(1 + \delta)L_i E_j L_j - \delta(1 + \delta)E_i L_i L_j + \delta^2 E_i L_i E_j L_j \end{aligned}$$

Hence

$$\begin{aligned} F_i F_j F_i &= (1 + \delta)^3 L_i L_j L_i - \delta(1 + \delta)^2 [L_i E_j L_j L_i + E_i L_i L_j L_i + L_i L_j E_i L_i] \\ &\quad + \delta^2(1 + \delta)E_i L_i E_j L_j L_i + \delta^2(1 + \delta)L_i E_j L_j E_i L_i + \delta^2(1 + \delta)E_i L_i L_j E_i L_i \\ &\quad - \delta^3 E_i L_i E_j L_j E_i L_i \end{aligned}$$

Using now (2.4), (2.5), (5.13) and (5.14) we get

$$\begin{aligned} F_i F_j F_i &= (1 + \delta)^3 L_i L_j L_i - \delta(1 + \delta)^2 [L_i E_j L_j L_i + E_i L_i L_j L_i + L_i L_j E_i L_i] \\ &\quad (2\delta^2(1 + \delta) - \delta^3) E_i E_j L_i L_j L_i + \delta^2(1 + \delta) E_i L_i L_j E_i L_i \end{aligned}$$

Now applying on the last term the relation (5.13) and using later (5.14), we get $E_i L_i L_j E_i L_i = L_i(E_i L_j E_i) L_i$, so

$$\begin{aligned} E_i L_i L_j E_i L_i &= L_i \left[E_i E_j L_j - \frac{1}{2}(E_i E_j - E_i) \right] L_i \\ &= E_i E_j L_i L_j L_i - \frac{1}{2} E_i E_j L_i^2 + \frac{1}{2} L_i E_i L_i \quad (\text{by using (5.14)}) \\ &= E_i E_j L_i L_j L_i - \frac{1}{2} E_i E_j L_i + \frac{1}{2} E_i L_i \quad (\text{by using (5.10) and (5.13)}) \end{aligned}$$

Then

$$\begin{aligned} F_i F_j F_i &= (1 + \delta)^3 L_i L_j L_i - \delta(1 + \delta)^2 [L_i E_j L_j L_i + E_i L_i L_j L_i + L_i L_j E_i L_i] \\ &\quad + (2\delta^3 + 3\delta^2) E_i E_j L_i L_j L_i - \delta^2(1 + \delta) \left[\frac{1}{2} E_i E_j L_i - \frac{1}{2} E_i L_i \right] \end{aligned}$$

On the other side, from (4)Lemma 5.2, we have

$$F_i + (u - 1)E_i F_i = (1 + \delta)L_i - \delta E_i L_i + (u - 1)E_i L_i = (1 + \delta)L_i - (u + 2)\delta E_i L_i$$

Therefore, the relation (5.16) is equivalent to

$$\begin{aligned} &(1 + \delta)^3 L_i L_j L_i - \delta(1 + \delta)^2 [L_i E_j L_j L_i + E_i L_i L_j L_i + L_i L_j E_i L_i] + (2\delta^3 + 3\delta^2) E_i E_j L_i L_j L_i \\ &= \frac{1}{(u + 1)^2} [(1 + \delta)L_i - (u + 2)\delta E_i L_i] + \delta^2(1 + \delta) \left[\frac{1}{2} E_i E_j L_i - \frac{1}{2} E_i L_i \right] \end{aligned}$$

which is reduced, after multiplication by $(u + 1)^2$, to

$$\begin{aligned} &\frac{8}{(u + 1)} L_i L_j L_i - 4\delta [L_i E_j L_j L_i + E_i L_i L_j L_i + L_i L_j E_i L_i] + (1 - u)^2 (2\delta + 3) E_i E_j L_i L_j L_i \\ &= (1 + \delta)L_i - (u + 2)\delta E_i L_i + (1 - u)^2 (1 + \delta) \left[\frac{1}{2} E_i E_j L_i - \frac{1}{2} E_i L_i \right] \end{aligned}$$

or equivalently

$$\begin{aligned} &\frac{8}{(u + 1)} L_i L_j L_i - 4\delta [L_i E_j L_j L_i + E_i L_i L_j L_i + L_i L_j E_i L_i] + (1 - u)^2 (2\delta + 3) E_i E_j L_i L_j L_i \\ &= (1 + \delta)L_i - 3\delta E_i L_i + (1 - u)\delta E_i E_j L_i \end{aligned}$$

Multiplying this last equation by $u + 1$ we obtain (5.16). \square

Remark 5.4. By taking $E_i = 1$ the elements L_i 's become f_i 's and the Theorem 5.3 and Theorem 5.1 become Theorem 1.3.

6. A LINEAR BASIS FOR PTL_n

By using essentially Theorems 5.1, 4.3 we shall construct a linear basis of PTL_n . Further we use also the following lemmas.

Lemma 6.1. *For all i, j such that $|i - j| = 1$, we have:*

- (1) $F_i E_j = T_i E_j T_i^{-1} F_i + \frac{1}{u+1} (E_j - T_i E_j T_i^{-1})$
- (2) $E_j F_i = F_i T_i E_j T_i^{-1} + \frac{1}{u+1} (E_j - T_i E_j T_i^{-1})$

Proof. It is enough to expand F_i in both side of the equality. \square

Lemma 6.2. *Any word in $1, F_1, \dots, F_{n-1}, E_1, \dots, E_{n-1}$ can be expressed as a K -linear combination of words in E_i 's and F_i 's having at most one F_{n-1}, E_{n-1} , or $F_{n-1}E_{n-1}$.*

Proof. It is a consequence of Proposition 1[1] and the fact that F_i is a linear expression of 1 and T_i . \square

Definition 6.3. *A word in F_1, \dots, F_{n-1} is called F -reduced (or simply reduced) if and only if has the form*

$$(F_{i_1} \cdots F_{j_1})(F_{i_2} \cdots F_{j_2}) \cdots (F_{i_k} \cdots F_{j_k}) \quad (6.1)$$

where $0 \leq k \leq n-1$ and

$$\begin{aligned} 1 \leq i_1 < i_2 < \cdots < i_k \leq n-1 \\ 1 \leq j_1 < j_2 < \cdots < j_k \leq n-1 \\ i_1 \geq i_2, i_2 \geq j_2, \dots, i_k \geq j_k \end{aligned}$$

Proposition 6.4. *Any word in $1, E_1, \dots, E_{n-1}, F_1, \dots, F_{n-1}$ may be written as K -linear combination of words in the form $E_I F$, where $I \in \mathcal{P}_n$ and F is F -reduced.*

Proof. We have adapted the proof of Proposition 2.8[5]. We will use induction on n . The assertion is clearly valid for $n = 2$. We assume now that the proposition is valid for n . Let W a word in $1, E_1, \dots, E_n, F_1, \dots, F_n$. By using Lemma 6.1 we can move the E_i 's appearing in W to the front position, obtaining in this way that W is a linear combination of words in the form $E_I W'$, where W' is a word in $1, F_1, \dots, F_n$. Thus, to prove the proposition it is enough to show that W' is a linear combination of words in the form desired. Now, if W' does not contain F_n then we are done. If W' contains F_n , according to Lemma 6.2, we have that W' is a linear combination of words in the form

$$W_1 R_n W_2$$

where $R_n = E_n, F_n$ or $E_n F_n$ and W_i are words in $1, E_1, \dots, E_{n-1}, F_1, \dots, F_{n-1}$. If $R_n = E_n$, according to Lemma 6.1, we can move R_n to the front position and then using the induction hypothesis we are done. Suppose $R_n = F_n$, we note that by induction hypothesis W_2 is a linear combination of words in the form

$$E_J V(F_n F_{n-1} \cdots F_{j_k})$$

where now $J \in \mathcal{P}_{n-1}$, V is a word reduced in $1, F_1, \dots, F_{n-2}$ (notice that $F_n F_{n-1} \cdots F_{j_k}$ could be empty). Hence W' is a linear combination of words of the form

$$W_1 F_n E_J V(F_n F_{n-1} \cdots F_{j_k})$$

Now, $F_n E_J = E_{s_n J} F_n$, so using (5.1) and (5.3) follows that W' can be written as a linear combination $(1 + \delta)N_1 - \delta N_2$ with $N_1 := E_{J'} V'(F_n F_{n-1} \dots F_{j_k})$ and $N_2 := E_{J'} V'(E_n F_n F_{n-1} \dots F_{j_k})$, where $J' \in \mathbf{P}_n$ and V' is a word in $1, F_1, \dots, F_{n-1}$. Again we note that in N_2 , E_n can move to the front position, so N_2 is in fact in the form of N_1 . Therefore, W' is a linear combination of words in the form

$$E_{J'} V'(F_n F_{n-1} \dots F_{j_k})$$

where $J' \in \mathbf{P}_n$ and V' is a word in $1, F_1, \dots, F_{n-1}$. Applying the induction hypothesis, on V' , we deduce that W' is a linear combination of words in the form $E_I F$, where $I \in \mathbf{P}_n$ and F has the form

$$F = (F_{i_1} \dots F_{j_1})(F_{i_2} \dots F_{j_2}) \dots (F_{i_k} \dots F_{j_k})$$

with i 's increasing and $i_l \geq j_l$, for all $1 \leq l \leq k$. Thus it remains to prove that in F 's the j 's can be taken increasing. Suppose $j_1 \geq j_2$, so

$$F = (F_{i_1} \dots F_{j_1+1})(F_{i_2} \dots (F_{j_1} F_{j_1+1} F_{j_1}) \dots F_{j_2}) \dots (F_{i_k} \dots F_{j_k})$$

Then, by using (5.7), we have $F = (u+1)^{-2} F_1 - (u+1)^{-1} \delta F_2$, where

$$F_1 := (F_{i_1} \dots F_{j_1+1})(F_{i_2} \dots F_{j_1} \dots F_{j_2}) \dots (F_{i_k} \dots F_{j_k})$$

and

$$F_2 := (F_{i_1} \dots F_{j_1+1})(F_{i_2} \dots (E_{j_1} F_{j_1}) \dots F_{j_2}) \dots F_{i_k} \dots F_{j_k})$$

Clearly (applying Lemma 6.1), E_{j_1} in F_2 can be moved to the front position. Therefore, by using an inductive argument we deduce that F can be expressed as a linear combination in the desired form. Hence W' can be written in the desired form. Thus, the proof is concluded. \square

Conjecture 6.5. *The set formed by the elements $E_I F$, where $I \in \mathbf{P}_n$ and F is reduced, is a linear basis of PTL_n . Hence the dimension of PTL_n is $b_n C_n$.*

7. MARKOV TRACE

For d a positive integer we denote $Y_{d,n} = Y_{d,n}(u)$ the Yokonuma–Hecke algebra, i.e. the algebra presented by braid generators g_1, \dots, g_{n-1} together with the framing generators t_1, \dots, t_n which satisfies the following defining relations: braids relation (of type A) among the g_i 's, $t_i t_j = t_j t_i$, $g_i t_j = t_{s_i(j)} g_i$ and

$$g_i^2 = 1 + (u-1)e_i(1+g_i)$$

where e_i is defined as

$$e_i := \frac{1}{d} \sum_{s=1}^d t_i^s t_{i+1}^{-s}$$

Proposition 7.1. *We have a natural algebra morphism $\psi : \mathcal{E}_n \mapsto Y_{d,n}$ defined through the mapping $T_i \mapsto g_i$ and $E_i \mapsto e_i$.*

Proof. According to Lemma 2.1[12] the defining relations of \mathcal{E}_n are satisfied by changing T_i by g_i and E_i by e_i . Hence the proof follows. \square

Theorem 7.2 (See [11]). *Let z, x_1, \dots, x_{d-1} be in \mathbb{C} . There exists a unique family of linear map $\{\text{tr}_n\}_n$ on inductive limit associated to the family $\{Y_{d,n}\}_n$ with values in \mathbb{C} satisfying the rules:*

$$\begin{aligned} \text{tr}_n(ab) &= \text{tr}_n(ba) \\ \text{tr}_n(1) &= 1 \\ \text{tr}_{n+1}(ag_n) &= z \text{tr}_n(a) && \text{for } a \in Y_{d,n} \\ \text{tr}_{n+1}(at_{n+1}^m) &= x_m \text{tr}_n(a) && \text{for } a \in Y_{d,n}, 1 \leq m \leq d-1. \end{aligned}$$

It is natural to consider the composition $\text{tr}_n \circ \psi$ which defines a Markov trace on \mathcal{E}_n . This supports the following conjecture.

Conjecture 7.3. [Aicardi, Juyumaya] *The algebra \mathcal{E}_n supports a Markov trace. I.e. for all $n \in \mathbb{N}$ we have a unique linear map $\rho_n : \mathcal{E}_n \longrightarrow K(A, B)$ such that for all $x, y \in \mathcal{E}_n$, we have:*

- (1) $\rho_n(1) = 1$
- (2) $\rho_n(xy) = \rho_n(yx)$
- (3) $\rho_{n+1}(xT_n) = \rho_{n+1}(xE_nT_n) = A\rho_n(x)$
- (4) $\rho_{n+1}(xE_n) = B\rho_n(x)$

where A and B are parameters.

Example 7.4. According to the rule (3) Conjecture 7.3 of ρ we have, $\rho(E_1T_1T_2T_1) = A\rho(E_1T_1^2)$. Now, $E_1T_1^2 = E_1(1 + (u-1)E_1(1+T_1)) = uE_1 + (u-1)E_1T_1$. So

$$\rho(E_1T_1T_2T_1) = A(uB + (u-1)A) = uAB + (u-1)A^2.$$

Assuming that Conjecture 7.3 is true, we are going to study when the Markov trace ρ_n passes to PTL_n . According to Corollary 4.6, studying the factorization of ρ_n to PTL_n is reduced to studying the values of ρ_n on the two-sided ideal generated by $E_iE_jT_{12}$. For this study we need the following lemmas.

- Lemma 7.5.**
- (1) $T_1T_{12} = [1 + (u-1)E_1]T_{12}$
 - (2) $T_2T_{12} = [1 + (u-1)E_2]T_{12}$
 - (3) $T_1T_2T_{12} = [1 + (u-1)E_1 + (u-1)E_{1,3} + (u-1)^2E_1E_2]T_{12}$
 - (4) $T_2T_1T_{12} = [1 + (u-1)E_2 + (u-1)E_{1,3} + (u-1)^2E_1E_2]T_{12}$
 - (5) $T_1T_2T_1T_{12} = [1 + (u-1)(E_1 + E_2 + E_{1,3}) + (u-1)^2(u+2)E_1E_2]T_{12}$

Proof. The proof of the statements results by expanding the left side and then using the defining relations of \mathcal{E}_n . As example we shall check the first statement:

$$\begin{aligned} T_1T_{12} &= T_1 + T_1^2 + T_1T_2 + T_1^2T_2 + T_1T_2T_1 + T_1^2T_2T_1 \\ &= T_1 + 1 + (u-1)E_1 + (u-1)E_1T_1 + T_1T_2 + T_2 \\ &\quad + (u-1)E_1T_2 + (u-1)E_1T_1T_2 + T_1T_2T_1 + T_2T_1 \\ &\quad + (u-1)E_1T_2T_1 + (u-1)E_1T_1T_2T_1 = T_{12} + (u-1)E_1T_{12}. \end{aligned}$$

□

- Lemma 7.6.**
- (1) $\rho_3(T_{12}) = (u+1)A^2 + 3A + (u-1)AB + 1$
 - (2) $\rho_3(E_{\{1,2,3\}}T_{12}) = (u+1)A^2 + (u+2)AB + B^2$
 - (3) $\rho_3(E_I T_{12}) = (u+1)A^2 + (u+1)AB + A + B$, for all $I \in \mathcal{P}(3)$ of cardinal 2.

Proof. The proof is only a routine of computations. We shall prove, as an example, the third claim. Suppose $I = \{\{1, 2\}, \{3\}\}$, hence $E_I = E_1$. Then, by linearity and using the example above we have

$$\rho_3(E_I T_{12}) = B + A + AB + A^2 + A^2 + uAB + (u - 1)A^2$$

Hence we have proved the claim. \square

Theorem 7.7. *The Markov trace $\rho_n : \mathcal{E}_n \longrightarrow K(A, B)$ passes to PTL_n if only if $A = -B$ or $A = -B/(1 + u)$.*

Proof. From Corollary 4.6 we have that ρ_n pass to PTL_n if only if $\rho_n(xE_1E_2T_{12}y) = 0$, for all $x, y \in \mathcal{E}_n$. Now, by linearity and trace properties of ρ_n follows that it is enough to study the conditions to have $\rho_n(xE_1E_2T_{12}) = 0$, for all x in a linear basis of \mathcal{E}_n . We consider now the basis \mathbb{S}_n of \mathcal{E}_n , see Theorem 2.4. Using the rules that define ρ_n we deduce that the computation of $\rho_n(xE_1E_2T_{12})$, for $x \in \mathbb{S}_n$, results in a $K(A, B)$ -linear combination of $\rho_3(zE_1E_2T_{12})$ with $z \in \mathbb{S}_3$. Now, z is of the form $E_I T_w$, with $w \in S_3$ and $I \in \mathbf{P}(\mathbf{3})$; since T_w commutes with E_1E_2 having in mind the Lemma 7.5 and the fact that E_1E_2 is the maxim element of $\mathbf{P}(\mathbf{3})$, we obtain that $zE_1E_2T_{12}$ is a K -scalar multiple of $E_1E_2T_{12}$. Therefore, $\rho_n(xE_1E_2T_{12}y) = 0$, for all $x, y \in \mathcal{E}_n$ is equivalent to have $\rho_3(E_1E_2T_{12}) = 0$. Now, from (2)Lemma 7.6, we have $\rho(E_1E_2T_{12}) = 0$ is equivalent to $(u + 1)A^2 + (u + 2)AB + B^2 = 0$, then $A = -B$ or $A = -B/(1 + u)$. \square

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